# CONCERNING JORDAN'S LINEAR GROUPS.

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## *Introduction.*

I PRESENT to the AMERICAN MATHEMATICAL SOCIETY to-day a continuation of the paper* presented last November, entitled *The group of holoedric transformation into itself of a given group.* To recall briefly: The given (abstract) group $G_n$ of order $n$ has the elements $s_1 =$ identity, $s_2, \ldots s_n$. The substitution-group $\Gamma^n$ of transformation of $G_n$ into itself is *the* substitution-group on the $n$ letters $s_1, \ldots s_n$ which leaves invariant the multiplication-table for $G_n$. Letters $s$ which are conjugate with one another under $\Gamma^n$ must as elements of $G_n$ have the same period. Thus, $s_1 =$ identity is invariant, and $\Gamma^n$ is really $\Gamma^{n-1}$ on the $n-1$ letters $s_2, \ldots s_n$.

We are to consider to-day the case that $\Gamma^{n-1}$ *is transitive on the* $n-1$ *letters* $s_2, \ldots s_n$. Then the $n-1$ elements $s_2, \ldots s_n$ of $G_n$ have the *same* period, which must then be a *prime* $p$. Hence $G_n$ has the order $n = p^\nu$. Every group $G_{n=p^\nu}$ has, in accordance with an important (Sylow's) theorem,† at least one element different from identity commutative with every element of the group. This property of the element may be read out of the multiplication-table for $G_{n=p^\nu}$, and is hence invariant under $\Gamma^{n-1}$. But $\Gamma^{n-1}$ is transitive on the $n-1$ letters $s_2, \ldots s_n$. Hence *every element of* $G_{n=p^\nu}$ *is commutative with every other element. Our given group* $G_n$ *is then the Abelian* $G_{p^\nu}$, or rather, omitting the ', $G_{p^n}$ *with* $n$ *generating elements, each of order* $p$, *and commutative with one another.* It will cause no confusion if we refer to it hereafter simply as *the Abelian* $G_{p^n}$.

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* *Bulletin of the American Mathematical Society*, ser. 2, vol. 1, pp. 61–66, Dec. 1894.

Mr. HÖLDER explained this notion of the group of holoedric transformations into itself of a given group, for use in his memoir: *Die Gruppen der Ordnungen* $p^3$, $pq^2$, $pqr$, $p^4$ (*Mathematische Annalen*, vol. 43, pp. 301–412; see pp. 313, 314), which bears the date March 28, 1893. We, however, hit on the notion independently of each other; see the foot-note (**) of p. 66 of my former paper.

† SYLOW: *Mathematische Annalen*, vol. 5, p. 588.

## § 1.

*The group $\Gamma^{p^n}_{\Omega(p^n)}$ of holoedric transformation into itself of the Abelian group $G_{p^n}$ is Jordan's linear homogeneous substitution-group of degree $p^n$, $LHG^{p^n}_{\Omega(p^n)}$.*

For the Abelian $G_{p^n}$ we take the $n$ generators

$$a_i \qquad (i=1, 2, \ldots n) \tag{1}$$

with the complete system of generating relations.

$$a_i^p = 1, \qquad a_i a_j = a_j a_i \qquad (i, j = 1, 2, \ldots n) \tag{2}$$

and have as the general element

$$s_K = s_{k_1, k_2, \ldots k_n} = a_1^{k_1} a_2^{k_2} \ldots a_n^{k_n} = \prod_i a_i^{k_i} \qquad (i=1, 2, \ldots n) \tag{3}$$

where the suffixes and exponents $k$ are integers taken modulo $p$, and where $K$ is a symbol standing for $(k_1, k_2, \ldots k_n)$.

The general multiplication equation is

$$s_{K_1} s_{K_2} = s_{K_3}, \quad s_{k_{11}, k_{12}, \ldots k_{1n}} \cdot s_{k_{21}, k_{22}, \ldots k_{2n}} = s_{k_{31}, k_{32}, \ldots k_{3n}}, \tag{4}$$

where

$$K_1 + K_2 = K_3, \quad k_{1i} + k_{2i} = k_{3i}. \qquad (i=1, 2 \ldots n) \tag{5}$$

It turns out that the general substitution $\sigma_G$ of $\Gamma^{p^n}_{\Omega(p^n)}$ re-replaces $s_X = s_{x_1, x_2, \ldots x_n}$ by $s_{X'} = s_{x'_1, x'_2, \ldots x'_n}$, where

$$X' = GX, \quad x'_i = \sum_{j=1}^{j=n} g_{ij} x_j \quad (|g_{ij}| \neq 0) \qquad (i, j = 1, 2, \ldots n), \tag{6}$$

where $G$ is a symbol for the matrix

$$G = (g_{ij}) \qquad (i, j = 1, 2, \ldots n) \tag{7}$$

whose elements $g_{ij}$ are integers taken modulo $p$. [To follow the customary notation we should write *congruences (modulo p)* everywhere instead of *equations.* But in group-theoretic applications such as the present, it is much better to breathe the *spirit* of the congruence once for all into the *definitions* of the symbols and operations.] Hence, indeed, $\Gamma^{p^n}_{\Omega(p^n)}$ is *Jordan's* linear homogeneous substitution-group * of degree $p^n$, $LHG^{p^n}_{\Omega(p^n)}$, of order †

$$\Omega(p^n) = (q^n - 1)(q^n - q)(q^n - q^2) \cdot \cdot (q^n - q^{n-1}). \tag{8}$$

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* JORDAN: *Traité des substitutions*, p. 92, 1870.
† JORDAN: loc. cit., p. 97.

This identification of the $\Gamma^{p^n}_{\Omega(p^n)}$ of the Abelian $G_{p^n}$ with the $LHG^{p^n}_{\Omega(p^n)}$ I obtain first by holding the $G_{p^n}$ as an *abstract* group; I omit the details of this identification. We may however take the $G_{p^n}$ *concretely* as the *regular* Abelian substitution-group $G^{p^n}_{p^n}$ on the $p^n$ letters $s_X = s_{x_1, x_2, \ldots x_n}$; the general element (3) $s_K = s_{k_1, k_2, \ldots k_n}$ then (4, 5) replaces $s_X$ by $s_{X'}$, where

$$X' = X + K, \quad x'_i = x_i + k_i \qquad (i = 1, 2, \ldots n). \tag{9}$$

We thus win direct contact with Mr. *Jordan's* work. The $G^{p^n}_{p^n}$ (9) is within the symmetric substitution-group on the $p^n$ letters $s_X$ self-conjugate under the linear non-homogeneous group $LG^{p^n}_{p^n\Omega(p^n)}$ of degree $p^n$ and of order $p^n\Omega(p^n)$, whose general substitution $\sigma_{G,K}$ replaces $s_X$ by $s_{X'}$, where

$$X' = GX + K, \quad x'_i = \sum_{j=1}^{j=n} g_{ij} x_j + k_i \quad (|g_{ij}| \neq 0) \qquad (i, j = 1, 2, \ldots n). \tag{10}$$

$\sigma_{G,K}$ replaces $s_{K_1}$, $s_{K_2}$, $s_{K_3}$ by $s_{K'_1}$, $s_{K'_2}$, $s_{K'_3}$, where

$$K'_1 = GK_1 + K, \quad K'_2 = GK_2 + K, \quad K'_3 = GK_3 + K,$$

so that

$$K'_1 + K'_2 - K'_3 = G(K_1 + K_2 - K_3) + K;$$

hence under $\sigma_{G,K}$ of the $LG^{p^n}_{p^n\Omega(p^n)}$ (10) a multiplication equation of the $G_{p^n}$ $s_{K_1} s_{K_2} = s_{K_3} = s_{K_1 + K_2}$ (4, 5) is preserved, that is,

$$s_{K'_1} s_{K'_2} = s_{K'_3} = s_{K'_1 + K'_2},$$

if and only if $K = (0) = (k_1, k_2, \ldots k_n) = (0, 0, \ldots 0)$, that is, if and only if the substitution $\sigma_{G,K}$ of the $LG^{p^n}_{p^n\Omega(p^n)}$ (10) is a substitution $\sigma_{G,0} = \sigma_G$ of the $LHG^{p^n}_{\Omega(p^n)}$ (6). We have then this (second) identification of the $\Gamma^{p^n}_{\Omega(p^n)}$ of the Abelian $G_{p^n}$ with the $LHG^{p^n}_{\Omega(p^n)}$.

The group $\Gamma^{p^n}_{\Omega(p^n)} \equiv LHG^{p^n}_{\Omega(p^n)}$ (6) is transitive on the $p^n - 1$ letters $s_X (X \neq (0))$. For $p = 2$ it is doubly transitive on the $p^n - 1 = 2^n - 1$ letters. For $p > 2$ it is simply transitive and imprimitive; the letter $s_X = s_{x_1, x_2, \ldots x_n}$ belongs to and by the ratios of its $n$ suffixes $\underset{\cdot\cdot}{X} = (x_1 : x_2 : \cdots : x_n)$ determines the system of imprimitivity containing the $q-1$ letters * $s_{lX} (l = 1, 2, \cdots \overline{p-1})$; in the $G_{p^n}$ the elements $s_{lX}$ and the identity $s_{0X} = s_{(0)}$ constitute the cyclic group $G_p\{s_X\}$ determined by $s_X$, say the $G_{p,X}$. Thus,

* $X = (x_1, \cdots x_n)$, $lX = (lx_1, \cdots lx_n)$.

the $\Gamma^{p^n}_{\Omega(p^n)} \equiv LHG^{p^n}_{\Omega(p^n)}$ permutes first the $(p^n-1)/(p-1)$ $G_p$ of $G_{p^n}$, and afterwards fixes the elements within the various groups. The self-conjugate sub-group which keeps every $G_p$ fixed is of order $p-1$:

$$(11) \qquad \{X' = lX,\ x'_i = lx_i (i = 1, 2, \cdots n)\},$$
$$(l = 1, 2, \cdots \overline{p-1}).$$

The quotient-group, which is a substitution-group on the $(p^n-1)/(p-1)$ $G_p$, has the order $\Omega(p^n)/(p-1)$. Analytically, it is the $LHG^{p^n}_{\Omega(p^n)}$ taken fractionally; that is, the linear fractional group† $LFG^{(p^n-1)/(p-1)}_{\Omega(p^n)/(p-1)}$, whose general substitution $\sigma_{\underset{\cdot\cdot}{G}}$ replaces the $G_{p,\underset{\cdot\cdot}{X}}$ by the $G_{p,\underset{\cdot\cdot}{X'}}$, where

$$(12) \qquad \underset{\cdot\cdot}{X'} = \underset{\cdot\cdot}{G}\,\underset{\cdot\cdot}{X},*$$

$$x'_1 : x'_2 : \cdots : x'_i : \cdots : x'_n = \sum_{j=1}^{j=n} g_{1j}x_j : \sum_{j=1}^{j=n} g_{2j}x_j : \cdots : \sum_{j=1}^{j=n} g_{ij}x_j : \cdots : \sum_{j=1}^{j=n} g_{nj}x_j$$

§ 2.

*Three tactical configurations:*

$$LCf[p^n],\ LHCf[p^n-1],\ LFCf[(p^n-1)/(p-1)]:$$

*connected with the Abelian $G_{p^n}$ are defining invariants respectively for the three linear groups:*

$$LG^{p^n}_{p^n\Omega(p^n)},\ LHG^{p^n \text{ or } p^n-1}_{\Omega(p^n)},\ LFG^{(p^n-1)/(p-1)}_{\Omega(p^n)/(p-1)}$$

The notion *configuration* I transfer to tactic from geometry†; for the proof and ultimate statement of the theorems about to be stated with utmost brevity, this notion must be used to its full content; to-day, however, the term *tactical configuration* shall be merely a name.

*The linear configuration $LCf[p^n]$ in $p^n$ letters.*

The $p^n$ letters of the $LCf[p^n]$ are the $p^n$ elements $s_X$ of the Abelian $G_{p^n}$. The $G_{p^n}$ contains $(p^n-1)/(p-1)$ sub-groups,

* JORDAN: loc. cit., p. 228. In my notation the two subscript dots (..) are the ratio dots (:), and are to call to mind that we may without changing anything replace $X = (x_1, \cdots x_n)$, $X' = (x'_1, \cdots x'_n)$, $G = (g_{ij})$ by $lX = (lx_1, \cdots lx_n)$, $l'X' = (l'x'_1, \cdots l'x'_n)$, $mG = (mg_{ij})$, respectively, where $l$, $l'$, $m$ are any integers taken modulo $p$, but $l \neq 0$, $l' \neq 0$, $m \neq 0$.

† See, for instance, REYE: *Das Problem der Configurationen* (*Acta Mathematica*, vol. 1, pp. 92–96, 1882).

$G_{p^{n-1}}$. With respect to each sub-group $G_{p^{n-1}}$ the $p^n$ elements $s_X$ of the $G_{p^n}$ are exhibited as a certain rectangular array of $p$ lines with $p^{n-1}$ elements in each line; the order of the lines and the order of the elements in each line are immaterial; one line contains the $p^{n-1}$ elements of the $G_{p^{n-1}}$ itself. We separate every array into its constituent lines, and have before us in the system of (unordered) $p(p^n-1)/(p-1)$ lines or combinations of $p^{n-1}$ letters each the linear configuration in $p^n$ letters, $LCf[p^n]$.

This $LCf[p^n]$ for $n \geqq 2$ defines, as the maximum substitution-group on the $p^n$ letters $s_X$ leaving it invariant, exactly the $LG^{p^n}_{p^n\Omega(p^n)}$ (§ 1 (10)).

*The linear homogeneous configuration $LHCf[p^n-1]$ in $p^n-1$ letters.*

The $p^n-1$ letters of the $LHCf[p^n-1]$ are the $p^n-1$ elements $s_X (X \neq (0))$ of the Abelian $G_{p^n}$, the identity $s_{(0)}$ excepted. The $LHCf[p^n-1]$ is obtained from the $LCf[p^n]$ by omitting every line or combination containing the discarded letter $s_{(0)}$. The $LHCf[p^n-1]$ consists, then, of a system of $p^n-1$ lines or combinations of $p^{n-1}$ letters each. This $LHCf[p^n-1]$ is tactically self-reciprocal,* that is, we can distribute a notation $s'_X$ to the $p^n-1$ lines in such a way that the $LHCf[p^n-1]$ on the $p^n-1$ letters $s_X$ as grouped by the $p^n-1$ lines $s'_X$ differs only in the priming (′) from the $LHCf[p^n-1]$ on the $p^n-1$ lines $s'_X$ as grouped by the $p^n-1$ letters $s_X$.

This $LHCf[p^n-1]$ for $n \geqq 2$ serves as a defining invariant for exactly the $LHG^{p^n \text{ or } p^{n-1}}_{\Omega(p^n)}$ (§ 1, (6)). The self-reciprocity of the $LHCf[p^n-1]$ establishes an holoedric isomorphism of the $LHG^{p^{n-1}}_{\Omega(p^n)}$ with itself. This isomorphism is (at least for $n \geqq 3$) *not** that arising from a transformation of the $LHG^{p^{n-1}}_{\Omega(p^n)}$ through one of its own elements.

*The linear fractional configuration $LFCf[(p^n-1)/(p-1)]$ on $(p^n-1)/(p-1)$ letters.*

The $(p^n-1)/(p-1)$ letters of the $LFCf[(p^n-1)/(p-1)]$ are the $(p^n-1)/(p-1)$ cyclic groups $G_{q;\,X}$ of the Abelian $G_{p^n}$. The $LFCf[(p^n-1)/(p-1)]$ is obtained from the $LCf[p^n]$ by

* Notice the particular case ($q=2$, $n=3$) in § 2 of my paper cited above. The $LHCf[2^3-1=7]$ and the $\Delta_7$ are, so to say, complementary. Indeed, for $q=2$, $n=$ any, the $LHCf[2^n-1]$ determines uniquely a $\Delta_{2^n-1}$, from which the $LHCf[2^n-1]$ is likewise uniquely determined. This $\Delta_{2^n-1}$ serves as a defining invariant for the $LHG^{2^n-1}_{\Omega(2^n)}$.

omitting every line not containing the identity letter $s_{(0)}$, that is, by retaining the lines corresponding to the $(p^n-1)/(p-1)$ sub-groups * $G_{p^{n-1}}$, and then in every such line by omitting the $s_{(0)}$ and replacing every set of $p-1$ letters $s_{lX}(l=1, 2, \ldots q-1)$ by the letter $G_{q;\,X}$. The $LFCf[(p^n-1)/(p-1)]$ on

$$(p^n-1)/(p-1)$$

letters consists then of a system of $(p^n-1)/(p-1)$ lines of $(p^{n-1}-1)/(p-1)$ letters each. This $LFCf[(p^n-1)/(p-1)]$ is tactically self-reciprocal.

This $LFCf[(p^n-1)/(p-1)]$ for $n \geqq 3$ serves as a defining invariant for exactly the $LFG^{(p^{n-1})/(p-1)}_{\Omega(p^n)/(p-1)}$ (§ 1, (12)). The self-reciprocity of the $LFCf[(p^n-1)/(p-1)]$ $(n \geqq 3)$ establishes an holoedric isomorphism of the $LFG^{(p^{n-1})/(p-1)}_{\Omega(p^n)/(p-1)}(n \geqq 3)$ with itself. This isomorphism is *not* that arising from a transformation of the $LFG^{(p^{n-1})/(p-1)}_{\Omega(p^n)/(p-1)}$ through one of its own elements.

In § 4 I give these various tactical configurations for certain low values of $p$ and $n$.

## § 3.

### *Utility of the Galois-field theory in the investigation of linear groups.*

The results given in § 2 depend for their proof largely upon the fact that *the group* $I^{p^{n-1}}_{\Omega(p^n)} \equiv LHG^{p^{n-1}}_{\Omega(p^n)}$ *contains a substitution* $\sigma_G$ *which permutes the* $p^n-1$ *letters* $s_X(X \neq (0))$ *in one cycle of period* $p^n-1$. Any such $\sigma_G$ answers the purpose. That one such exists we know from the Galois-field theory.†

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* This linear fractional configuration might also be called the *sub-group configuration of the Abelian* $G_{p^n}$.

† GALOIS: *Sur la théorie des nombres* (*Bulletin des Sciences Mathématiques* de M. Ferussac, vol. 13, p. 428, 1830; reprinted, *Journal de Mathématiques pures et appliquées*, vol. 11, pp. 398–407, 1846.)

SERRET: *Algèbre supérieure*, fifth edition, vol. 2, pp. 122–189.

JORDAN: *Traité des substitutions*, pp. 14–18.

MOORE: *A doubly infinite system of simple groups* (§ 3 is an abstract formulation of the Galois-field theory). (*Proceedings of the Chicago Congress of Mathematics;* in abstract, *Bulletin of the New York Mathematical Society*, vol. 3, Dec. 1893.)

Addendum of Oct. 15, 1895. I have found within a week that MATHIEU in Chapter III, pp. 275–304, of his *Mémoire sur l'étude des fonctions de plusieurs quantités, sur la manière de les former et sur les substitutions qui les laissent invariables* (*Journal de Mathématiques pures et appliquées*, ser. 2, vol. 6, pp. 241–323, 1861), working from the Galois-field standpoint, defines and investigates two substitution-groups, which are (otherwise expressed) the groups $LG^{p^n}_{p^n\Omega(p^n)}$ and $LHG^{p^n}_{\Omega(p^n)}$. This seems to be the source from which Mr. Jordan's linear groups (1870) were drawn. Mathieu gives two rational integral functions of the $p^n$

Suppose that the $p^n$ marks $\xi$ of the Galois-field $GF[p^n]$ of order $p^n$ are exhibited explicitly in terms of $n$ linearly independent marks $\eta_1, \eta_2 \cdots \eta_n$ in the form

$$\xi = \sum_{i=1}^{i=n} x_i \eta_i \tag{1}$$

where the $x_i$'s are integral marks, or integers taken modulo $p$. We make through the $X=(x_1, \cdots x_n)$ a 1.1 correspondence between the letters $s_X$ and the marks $\xi$. In fact the $GF[p^n]$ quâ additive-group is a concrete Abelian $G_{p^n}$. Now in the $GF[p^n]$ additions are invariant under the multiplication-substitution $\sigma_\gamma$ on the $p^n-1$ marks $\xi$ $(\xi \neq 0)$,

$$\xi' = \gamma\xi, \ (\gamma \neq 0), \tag{2}$$

that is, when every mark of the field is multiplied by the same mark $\gamma$. Hence this $\sigma_\gamma$ interpreted on the $s_X$ is a substitution $\sigma_G$ of the $\Gamma^{p^n-1}_{\Omega(p^n)} \equiv LHG^{p^n-1}_{\Omega(p^n)}$. If $\gamma$ is a primitive root of the $GF[p^n]$, $\sigma_\gamma$ permutes the $p^n-1$ marks $\xi(\xi \neq 0)$ in one cycle, and, similarly, $\sigma_G$ permutes the $p^n-1$ letters $s_X(X \neq (0))$ in one cycle, and is then the substitution sought.

The results of § 2 constitute for the linear groups sweeping generalizations of Mr. *Noether's* definition* of the group $\Gamma^7_{168}$ by the triple system $\Delta_7$ in seven letters.

## § 4.

*Tables*†§ *of the tactical configurations:*
$LCf[p^n]$, $LHCf[p^n-1]$, $LFCf[(p^n-1)/(p-1)]$,
*for cases* $p^n = 2^2, 3^2, 5^2, 7^2, 11^2; \ 2^3, 3^3, 5^3, 7^3; \ 2^4, 3^4, 5^4; \ 2^5; \ 2^6.$

The table for a particular case $[p^n]$ gives first a primitive root $\gamma$ of the Galois-field $GF[p^n]$ and its fundamental equation

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letters $s_X$, each of which serves as defining invariant for the $LG^{p^n}_{p^n(p\Omega^n)}$. These functions are closely related to our $LCf[p^n]$. In explaining my researches in detail in a subsequent paper I shall point out the exact points of contact with MATHIEU's results.

It should be added that several weeks ago Mr. Dickson and I came upon a substitution-group on the $p^n$ marks of the $GF[p^n]$ which Mr. Dickson then identified as another expression of Mr. Jordan's $LHG^{p^n}_{\Omega(p^n)}$; this was exactly MATHIEU's expression of the group.

* See § 2 of my paper cited above.

† The theory of the linear fractional configuration I introduced in my course *Groups*, during the last spring quarter at the University of Chicago, and in connection with the members of that course, Messrs. Brown, Dickson, Joffe, and Slaught, worked up the linear fractional configurations § for the cases given above, except $p^n = 2^6$. I take this opportunity to thank them for their coöperation, and especially Mr. Dickson, who quite recently completed the tables as given above.

§ I add the tables for $p^n = 2^2, 3^2, 5^2, 7^2, 11^2$, whose linear *fractional* configurations are trivial. Sept. 10, 1895.

of degree $n$. The $p^n$ elements of the abstract $G_{p^n}$ have the *index-notation* * derived from the $p^n$ marks of the $GF[p^n]$ (as a concrete $G_{p^n}$, § 3): mark $\xi = 0$, index $*$; mark $\xi \neq 0$, $\xi = \gamma^i$, index $i$ $(i = 0, 1, \ldots \overline{p^n - 1})$; $i$ is an integer taken modulo $p^n - 1$.

The $LCf[p^n]$ consists (§ 2) of the lines found in certain $(p^n - 1)/(p - 1)$ arrays. Each array has $p$ lines; each line has $p^{n-1}$ indices. Only the first array is given; the others are obtained from it by repeated applications of the cyclical substitution

$$i' = i + 1, \qquad (i = 0, 1, \ldots \overline{p^n - 1}),$$

which leaves $*$ fixed. The first line of the first array is the additive sub-group $G_{p^{n-1}}$ of the $GF[p^n]$ quâ additive $G_{p^n}$, which contains the $n - 1$ marks $\gamma^0, \gamma^1, \ldots \gamma^{n-2}$. The second line is obtained by adding the *mark* $\gamma^{n-1}$ to the *marks* of the first line. Of course the lines one and two must be expressed in the index-notation. The following lines are derived from the second at once by repeated additions of $(p^n - 1)/(p - 1)$ to the *indices* of the second line.

The $LHCf[p^n - 1]$ and the $LFCf[(p^n - 1)/(p - 1)]$ are easily derived from the $LCf[p^n]$ (§ 2). The $LCf[p^n]$ and the $LHCf[p^n - 1]$ are tabulated together.

## TABLES.†

$[p^n = 2^2]$ $GF[2^2]$ Primitive root $\gamma$ where $\gamma^2 = 1 + \gamma$.
$LCf[2^2]$ 4 indices $*$ 0 1 2.
$LHCf[2^2 - 1]$ 3 indices 0 1 2.

$[* \; 0]_2 \quad [1 \; 2]_2$ §

$[p^n = 3^2]$ $GF[3^2]$ Primitive root $\gamma$ where $\gamma^2 = 1 + 2\gamma$.
$LCf[3^2]$ 9 indices $*$ 0 1 . . 7.
$LHCf[3^2 - 1]$ 8 indices 0 1 . . 7.

$[* \; 0 \; 4]_3 \quad [1 \; 6 \; 7]_3 \quad [2 \; 3 \; 5]_3$

$[p^n = 5^2]$ $GF[5^2]$ Primitive root $\gamma$ where $\gamma^2 = 2 + 2\gamma$.
$LCf[5^2]$ 25 indices $*$ 0 1 . . 23.
$LHCf[5^2 - 1]$ 24 indices 0 1 . . 23.

$[* \; 0 \; 6 \; 12 \; 18]_5 \quad [1 \; 3 \; 4 \; 8 \; 17]_5 \quad [7 \; 9 \; 10 \; 14 \; 23]_5 \quad [5 \; 13 \; 15 \; 16 \; 20]_5$
$[2 \; 11 \; 19 \; 21 \; 22]_5$

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* The $s_x$ notation for the elements can be recovered if necessary.

† The $LFCf\left[\frac{p^n - 1}{p - 1}\right]$ for $n = 2$ is trivial and hence is not tabulated.

§ Every *line* [ ] has a suffix indicating the number of *indices* lying within.

$[p^n = 7^2]$ $GF[7^2]$ Primitive root $\gamma$ where $\gamma^2 = 2 + 2\gamma$.
$LCf[7^2]$ 49 indices $*$ 0 1 . . 47.
$LHCf[7^2 - 1]$ 48 indices 0 1 . . 47.

$[* \ 0\ 8\ 16\ 24\ 32\ 40]_7$
$[1\ 6\ 18\ 20\ 21\ 27\ 31]_7$ $[9\ 14\ 26\ 28\ 29\ 35\ 39]_7$ $[17\ 22\ 34\ 36\ 37\ 43\ 47]_7$
$[3\ 7\ 25\ 30\ 42\ 44\ 45]_7$ $[2\ 4\ 5\ 11\ 15\ 33\ 38]_7$ $[10\ 12\ 13\ 19\ 23\ 41\ 46]_7$

$[p^n = 11^2]$ $GF[11^2]$ Primitive root $\gamma$ where $\gamma^2 = 9 + 4\gamma$.
$LCf[11^2]$ 121 indices $*$ 0 1 . . 119.
$LHCf[11^2 - 1]$ 120 indices 0 1 . . 119.

$[* \ 0\ 12\ 24\ 36\ 48\ 60\ 72\ 84\ 96\ 108]_{11}$
$[1\ 27\ 55\ 58\ 65\ 66\ 71\ 80\ 98\ 100\ 117]_{11}$
[Second line] + 12; 24; 36; 48; 60; 72; 84; 96; 108 = the respective remaining lines.

$[p^n = 2^3]$ $GF[2^3]$ Primitive root $\gamma$ where $\gamma^3 = 1 + \gamma$.
$LCf[2^3]$ 8 indices $*$, 0, 1, . . 7.
$LHCf[2^3 - 1]$ 7 indices 0, 1, . . 7.

$[* \ 0\ 1\ 3]_4$ $[2\ 4\ 5\ 6]_4$

$LFCf[(2^3 - 1)/(2 - 1)]$ 7 indices 0, 1, . . 7.

$[0\ 1\ 3]_3$

$[p^n = 3^3]$ $GF[3^3]$ Primitive root $\gamma$ where $\gamma^3 = 2 + \gamma$.
$LCf[3^3]$ 27 indices $*$, 0, 1, . . 25.
$LHCf[3^3 - 1]$ 26 indices 0, 1, . . 25.

$[* \ 0\ 1\ 3\ 9\ 13\ 14\ 16\ 22]_9$
$[2\ 4\ 6\ 7\ 10\ 11\ 12\ 18\ 21]_9$
$[15\ 17\ 19\ 20\ 23\ 24\ 25\ 5\ 8]_9$

$LFCf[(3^3 - 1)/(3 - 1)]$ 13 indices 0, 1, . . 12.

$[0\ 1\ 3\ 9]_4$

$[p^n = 5^3]$ $GF[5^3]$ Primitive root $\gamma$ where $\gamma^3 = 3 + 2\gamma$.
$LCf[5^3]$ 125 indices $*$, 0, 1, . . 123.
$LHCf[5^3 - 1]$ 124 indices 0, 1, . . 123.

$[* \ 0\ 1\ 3\ 10\ 14\ 26\ 31\ 32\ 34\ 41\ 45\ 57\ 62\ 63\ 65\ 72\ 76\ 88\ 93\ 94\ 96\ 103\ 107\ 119]_{25}$
$[2\ 9\ 13\ 15\ 28\ 29\ 30\ 35\ 38\ 39\ 48\ 53\ 56\ 68\ 80\ 82\ 98\ 104\ 105\ 109\ 112\ 114\ 116\ 117\ 120]_{25}$
[Second line] + 31; 62; 93 = the respective remaining lines.

$LFCf[(5^3 - 1)/(5 - 1)]$ 31 indices 0, 1, . . 30.

$[0\ 1\ 3\ 10\ 14\ 26]_6$

$[p^n = 7^3]$ $GF[7^3]$ Primitive root $\gamma$ where $\gamma^3 = 5 + \gamma$.
$LCf[7^3]$ 343 indices $*$, 0, 1, . . 341.
$LHCf[7^3 - 1]$ 342 indices 0, 1, . . 341.

[$*$ 0 1 3 13 32 36 43 52 57 58 60 70 89 93 100 109 114 115 117 127 146 150 157 166 171 172 174 184 203 207 214 223 228 229 231 241 260 264 271 280 285 286 288 298 317 321 328 337]$_{49}$

[2 4 6 9 14 16 26 33 35 41 44 45 46 50 56 64 75 78 82 86 99 133 134 142 148 168 181 186 194 195 201 202 218 219 222 240 245 265 267 268 277 281 283 290 293 296 307 312 323]$_{49}$

[Second line] + 57; 114; 171; 228; 285 = the respective remaining lines.

$LFCf[(7^3 - 1)/(7 - 1)]$ 57 indices 0, 1, . . 56.

[0 1 3 13 32 36 43 52]$_8$

$[p^n = 2^4]$ $GF[2^4]$ Primitive root $\gamma$ where $\gamma^4 = 1 + \gamma$.
$LCf[2^4]$ 16 indices $*$, 0, 1, . . 14.
$LHCf[2^4 - 1]$ 15 indices 0, 1, . . 14.

[$*$ 0 1 2 4 5 8 10]$_8$ [3 6 7 9 11 12 13 14]$_8$

$LFCf[(2^4 - 1)/(2 - 1)]$ 15 indices 0, 1, . . 14.

[0 1 2 4 5 8 10]$_7$

$[p^n = 3^4]$ $GF[3^4]$ Primitive root $\gamma$ where $\gamma^4 = 1 + \gamma + 2\gamma^2 + 2\gamma^3$.
$LCf[3^4]$ 81 indices $*$, 0, 1, . . 79.
$LHCf[3^4 - 1]$ 80 indices 0, 1, . . 79.

[$*$ 0 1 2 5 12 18 22 24 26 27 29 32 33 40 41 42 45 52 58 62 64 66 67 69 72 73]$_{27}$

[3 7 15 17 20 21 30 31 37 38 44 46 48 49 50 51 53 54 56 59 63 65 68 74 75 76 79]$_{27}$

[Second line] + 40.

$LFCf[(3^4 - 1)/(3 - 1)]$ 40 indices 0, 1, . . 39.

[0 1 2 5 12 18 22 24 26 27 29 32 33]$_{13}$

$[p^n = 5^4]$ $GF[5^4]$ Primitive root $\gamma$ where $\gamma^4 = 2 + \gamma + \gamma^2$.
$LCf[5^4]$ 625 indices $*$, 0, 1, . . 623.
$LHCf[5^4 - 1]$ 624 indices 0, 1, . . 623.

[$*$ {0 1 2 7 18 19 23 36 43 44 46 47 55 57 61 64 70 76 77 84 86 89 92 94 96 108 119 122 143 148 152} + 0, 156, 312, 468]$_{125}$

[3 4 5 10 17 21 22 29 31 37 39 41 42 59 63 68 74 88 95 99 104 107 109 110 127 130 134 141 146 153 162 165 168 169 181 186 189 190 191 194 196 207 208 216 218 221 222 225 229 231 237 239 241 261 262 269 270 271 272 276 277 279 281 285 287 288 289 291 294 300 305 306 323 328 332 336 338 361 362 365 366 379 383 390 399 402 405 410 412 413 415 424 430 438 440 451 454 467 476 482 483 495 496 500 513 516 524 526 540 547 548 550 559 565 570 579 585 592 604 605 608 613 615 619 622]$_{125}$

[Second line] + 156; 312; 468 = the respective remaining lines.

$LFCf[(5^4 - 1)/(5 - 1)]$ 156 indices 0, 1, . . 155.

[The { } of first line above]$_{31}$

$[p^n = 2^5]$ $GF[2^5]$ Primitive root $\gamma$ where $\gamma^5 = 1+\gamma+\gamma^2+\gamma^3$.
$LCf[2^5]$ 32 indices $*$, 0, 1, . . 30.
$LHCf[2^5-1]$ 31 indices 0, 1, . . 30.

$[* \ 0 \ 1 \ 2 \ 3 \ 5 \ 8 \ 10 \ 12 \ 13 \ 14 \ 18 \ 24 \ 25 \ 27 \ 28]_{16}$
$[4 \ 6 \ 7 \ 9 \ 11 \ 15 \ 16 \ 17 \ 19 \ 20 \ 21 \ 22 \ 23 \ 26 \ 29 \ 30]_{16}$

$LFCf[(2^5-1)/(2-1)]$ 31 indices 0, 1, . . 30.

$[0 \ 1 \ 2 \ 3 \ 5 \ 8 \ 10 \ 12 \ 13 \ 14 \ 18 \ 24 \ 25 \ 27 \ 28]_{15}$

$[p^n = 2^6]$ $GF[2^6]$ Primitive root $\gamma$ where $\gamma^6 = 1+\gamma+\gamma^3+\gamma^4$.
$LCf[2^6]$ 64 indices $*$, 0, 1, . . 62.
$LHCf[2^6-1]$ 63 indices 0, 1, . . 62.

$[* \ 0 \ 1 \ 2 \ 3 \ 4 \ 6 \ 13 \ 14 \ 16 \ 18 \ 20 \ 21 \ 22 \ 25 \ 26 \ 31 \ 35 \ 37 \ 40 \ 42 \ 43 \ 46 \ 49$
$50 \ 51 \ 53 \ 54 \ 56 \ 57 \ 58 \ 59]_{32}$
$[5 \ 7 \ 8 \ 9 \ 10 \ 11 \ 12 \ 15 \ 17 \ 19 \ 23 \ 24 \ 27 \ 28 \ 29 \ 30 \ 32 \ 33 \ 34 \ 36 \ 38 \ 39 \ 41$
$44 \ 45 \ 47 \ 48 \ 52 \ 55 \ 60 \ 61 \ 62]_{32}$

$LFCf[(2^6-1)/(2-1)]$ 63 indices 0, 1, . . 62.

[First line above, omitting the $*$]$_{31}$

THE UNIVERSITY OF CHICAGO,
*August* 25, 1895.

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## ELEMENTARY PROOF OF THE QUATERNION ASSOCIATIVE PRINCIPLE.*

BY PROFESSOR ARTHUR S. HATHAWAY.

THE variety of demonstrations that Hamilton has given of the associative principle of quaternion multiplication, and the remarks that he has made upon such demonstrations, show that he considered an elementary proof of this principle as very desirable. Only two of Hamilton's proofs have been generally employed by subsequent writers—the direct proof by spherical conics, and the indirect one depending upon the assumed laws of $i, j, k$—and the proof that he considered the most elementary has been entirely ignored, probably because of its deviation from fundamental ideas. On page 297 of the *Elements*, Hamilton calls attention to another method, as follows: "The *associative principle* of multiplication may also be proved without the distributive principle, by certain considerations of *rotations of a system*, on which we cannot enter here."

It is, of course, easy to see that such a proof is possible; but the details of it could not have presented themselves to Hamilton in an elementary form, or he would have seen that it

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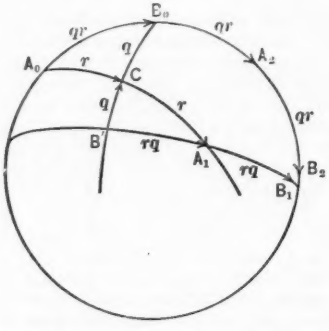
* Presented to the AMERICAN MATHEMATICAL SOCIETY August 28, 1895.

was just the demonstration for which he was looking, simple in character, and direct in its application.

The foundation consists of a few simple data with regard to the rotation of a sphere about its diameters, and a lemma that immediately identifies versor multiplication with the composition of rotations.

By the rotation, 2 *arc q*, we mean the rotation whose *equator, angular magnitude,* and *direction,* are given by double the arc that gives the *plane, angular magnitude,* and *direction,* of the turning or versor operation of $q$. The axis of $q$ is therefore the axis of the rotation; and as *arc q* is not greater than a semicircumference, so 2 *arc q* is not greater than a circumference. There is not more than one axis around which the sphere may be rotated from one position to another, since any diameter that is oblique to one axis returns to its original position only for entire revolutions of the sphere, and cannot therefore be another axis of rotation. Hence there is not more than one equator pertaining to the two positions of the sphere, and all arcs that specify rotations between those positions must lie upon the same great circle and be equal in magnitude and direction to one or the other of two oppositely directed arcs whose combined lengths make a circumference. The halves of any two such arcs are therefore either equal and concurrent, or supplementary and opposite.

*Lemma. The rotation 2 arc r, followed by the rotation 2 arc q, produces the same displacement of the sphere as the single rotation 2 arc qr.*




For let $C$ be a point of intersection of the great circles of $r$ and $q$, and lay off the given arcs, *arc* $r = A_0C = CA_1$, *arc* $q = B'C = CB_0$; and also construct *arc* $rq = B'A_1 = A_1B_1$, and *arc* $qr = A_0B_0 = B_0A_2 = A_2B_2$. By construction, the spherical tri-

angles $A_0B_0C$, $A_1B'C$, are equal; and therefore the arcs $A_0B_0$, $A_1B_1$, are equal and equally inclined to the arc $A_0A_1$, and the continuous arcs $B'A_1$, $A_1B_1$, are equal to the continuous arcs $B_0A_2$, $A_2B_2$, and these two continuous arcs are equally inclined to $B'B_0$. Hence, if a marked arc $AB$ of the sphere initially coincide with $A_0B_0$, then the rotation of the sphere through $2\ arc\ r = A_0A_1$, brings $AB$ into the position $A_1B_1$, and the next rotation through $2\ arc\ q = B'B_0$, brings $AB$ into the position $A_2B_2$. From the figure it is obvious that the position $A_2B_2$ is reached from the position $A_0B_0$ by the single rotation $2\ arc\ qr = A_0A_2$. Q.E.D.

This lemma shows that the successive rotations, $2\ arc\ r$, $2\ arc\ q$, $2\ arc\ p$, produce the same displacement of the sphere as either of the single rotations, $2\ arc\ p(qr)$, or $2\,arc\,(pq)\,r$ (according as we begin by compounding the first and second, or the second and third, rotations into their equivalent single rotations). Hence, $arc\ p(qr)$ and $arc\,(pq)\,r$ are equal and concurrent, or supplementary and opposite arcs of the same great circle. We may rely upon a diagram to show the currency of these two arcs, and therefore their complete versor equality, or we may prove the same by the principles of continuity, as follows: When $p$, $q$, $r$, change continuously in value, their arcs and the arcs of their products also change continuously in circle length and direction. Hence, if $arc\ p(qr)$ and $arc\ (pq)\,r$ are equal and concurrent for particular values of $p$, $q$, $r$ (as is obviously the case), they must continue so for all values of $p$, $q$, $r$, since an indefinitely small change in each arc cannot change them from equal concurrent arcs to the only other possible case of supplementary opposite arcs.

So in general, if the sphere receive the successive rotations, $2\,arc\,r$, $2\,arc\,q$, $2\,arc\,p$, . . . , then its displacement is the same as by the single rotation, $2\,arc\,(\ldots pqr)$, in whatever way the factors of this product are associated to form such product. Therefore the arcs of any two such products are either equal and concurrent, or supplementary and opposite arcs of the same great circle; and the principles of continuity, and the special case in which each arc begins where the preceding arc ends, show that the first-named relation is the true one. Since the tensors of all such products are also equal, therefore all such products, formed by different associations of the same factors in the same order, are equal.

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# MORAL VALUES.*

BY R. HENDERSON.

THE question of moral values in connection with the Theory of Probability is one which has given rise to great diversity of opinion among mathematicians. In Bertrand's *Calcul des Probabilités*, which is, perhaps, the most important work of recent years on the subject of Probability, it is dismissed with contempt, while in some other works considerable space is devoted to it.

Perhaps one of the reasons for the disrepute in which some hold the theory of moral values, is the fact that almost all who have taken up the subject from a mathematical standpoint have adhered to Bernoulli's or some other equally rigid hypothesis. Bernoulli's hypothesis is that the increase in moral value, arising from an indefinitely small increase in wealth, varies directly as that increase and inversely as the wealth already in possession, or, in other words, denoting the moral value of a wealth $x$ by the symbol $f(x)$, that $\frac{df(x)}{dx} = \frac{a}{x}$, where $a$ is a constant, or, integrating this, that $f(x) = a \log x + c$, $c$ being another arbitrary constant. The form of $f(x)$ adopted by Cramer was $f(x) = ax^{\frac{1}{2}}$. It would seem that many, being unwilling to admit these hypotheses, have been inclined to ridicule the whole subject, and have been to a certain extent blind to the basis of fact on which it is founded.

Now it has long seemed to me that Bernoulli did not intend his hypothesis as an absolute statement of the law of moral values, but rather as merely a particular case of a more general hypothesis, however definite or indefinite in his mind that general hypothesis may have been. And indeed writers on the subject of Economics, to whose department the question of the admissibility, or the contrary, of the hypothesis properly belongs, have adopted, though I cannot say they have originated, a more general hypothesis treated of below.

The earliest statement of this general hypothesis which I have seen is in the introduction to Laplace's *Théorie Analytique des Probabilités*, page xxii of the edition of 1847, at which place there occurs a passage of which the following is a free translation:

"The disadvantage of gambling and of exposing to the same risk all goods to be shipped, and all similar results indicated by good sense, subsist, whatever function the moral value may be of the material fortune of each individual. It is sufficient that the ratio of the increase of that function to

* Presented to the AMERICAN MATHEMATICAL SOCIETY, October 26, 1895.

the increase of the material fortune diminish as this latter increases."

Professor Marshall in his *Principles of Economics*, Appendix, Note ix, by a course of reasoning similar to that below, proves the disadvantage of gambling, provided $f(x)$ is amenable to expansion in Taylor's series in the form which he uses, that is, provided $\frac{d^2f(x)}{dx^2}$ is continuous and finite; but I have not been able to find anywhere a complete proof of the truth of these statements of Laplace, and it might be interesting to examine them analytically.

Denoting, as above, the moral value of a material fortune of amount $x$, by $f(x)$, Laplace's statement is equivalent to saying that it is sufficient that $\frac{df(x)}{dx}$ should diminish as $x$ increases, or, in other words, that $\frac{d^2f(x)}{dx^2}$ should be always negative, but not necessarily finite nor continuous. I shall introduce the extra assumption that $\frac{df(x)}{dx}$ is always finite which enables me to use the following expression for $f(x+h)$, the source of which I cannot at this moment recollect.

$$\begin{aligned} f(x+h) &= f(x) + \int_0^h f'(x+h-z)\,dz \\ &= f(x) + hf'(x) + \int_0^h zf''(x+h-z)\,dz \\ &= f(x) + hf'(x) + \theta h \int_0^h f''(x+h-z)\,dz \\ &= f(x) + hf'(x) + \theta h\{f'(x+h) - f'(x)\}, \end{aligned}$$

where $\theta$ is a proper fraction, since $f''(x+h-z)$ retains the same sign throughout. And since that sign is negative, the sign of $\int_0^h f''(x+h-z)\,dz$ is necessarily different from that of $h$, and consequently the last term $\theta h\{f'(x+h) - f'(x)\}$ is essentially negative. These results hold subject to the condition above stated that $f'(x)$ should be always finite.

This equation then may be expressed in the form

$$f(x+h) < f(x) + hf'(x)$$

for all values of $h$. From this inequality it is easy to deduce the following theorem:

*Theorem.* If $f(x)$ be a function of $x$ such that $f'(x)$ is finite and $f''(x)$ is negative for all values of $x$, then for all positive values of $l$ and $m$ we have

$$\frac{lf(a) + mf(b)}{l+m} < f\left(\frac{la+mb}{l+m}\right).$$

For, let $\frac{la+mb}{l+m}=x$ and $\frac{b-a}{l+m}=h$,

then $a=x-mh$ and $b=x+lh$,

and $\frac{lf(a)+mf(b)}{l+m}=\frac{l}{l+m}f(x-mh)+\frac{m}{l+m}f(x+lh)$

$$<\frac{l}{l+m}\{f(x)-mhf'(x)\}+\frac{m}{l+m}\{f(x)+lhf'(x)\},$$

$$i.e. \quad <f(x),$$

$$i.e. \quad <f\left(\frac{la+mb}{l+m}\right).$$

*Corollary.* Applying this theorem successively, we get

$$\frac{lf(a)+mf(b)+nf(c)+\cdots}{l+m+n+\cdots}<f\left(\frac{la+mb+nc+\cdots}{l+m+n+\cdots}\right),$$

where $l$, $m$, $n$, etc., are all positive.

Supposing now that the moral value of a material fortune $x$ may be represented by $f(x)$, a function which satisfies the above conditions, we have the following results:

(1) *A bet at fair odds is morally disadvantageous.*

For, let the man's material fortune before making the bet be $a$, and let $x$ be staked against $y$ in a fair wager.

Then the probability of winning is $\frac{x}{x+y}$, and of losing is $\frac{y}{x+y}$. Assuming then that the moral value of his expectation is equal to the sum of the moral values of the various amounts of which he may ultimately stand possessed multiplied respectively by their probabilities, we have it equal, after the bet is made, to

$$\frac{x}{x+y}f(a+y)+\frac{y}{x+y}f(a-x),$$

which by the theorem is less than

$$f\left(\frac{x}{x+y}(a+y)+\frac{y}{x+y}(a-x)\right),$$

or $f(a)$ the moral value at first.

(2) *If a merchant expects a shipment, the moral value of his expectation is greater if the shipment be divided than if it be shipped in one vessel only, the probability of each vessel arriving being the same.*

Let $v$ be the merchant's wealth in possession, $s$ the value of the shipment, and $p$ the probability of each vessel arriving, and let $(1-p)=q$.

Then if it be shipped in one vessel, the moral value of his expectation is $pf(v+s)+qf(v)$.

But if it be shipped in $t$ different vessels,

The probability of

all arriving is $p^t$, when his wealth will be $(v+s)$.

$t-1$ " $tp^{t-1}q$, " $\left(v+\frac{t-1}{t}s\right)$.

$t-2$ " $\frac{t(t-1)}{1\cdot 2}p^{t-2}q^2$, " $\left(v+\frac{t-2}{t}s\right)$.

etc.

Therefore the moral value of his expectation is

$$p^tf(v+s)+tp^{t-1}qf\left(v+\frac{t-1}{t}s\right)+\frac{t(t-1)}{1\cdot 2}f\left(v+\frac{t-2}{t}s\right)+\cdots+q^tf(v),$$

$$>p^tf(v+s)+\{(t-1)p^{t-1}qf(v+s)+p^{t-1}qf(v)\}+\left\{\frac{(t-1)(t-2)}{1\cdot 2}f(v+s)+\frac{t-1}{1}f(v)\right\}+\cdots+q^tf(v),$$

$$i.e.\ >p(p+q)^{t-1}f(v+s)+q(p+q)^{t-1}f(v),$$

$$i.e.\ >pf(v+s)+qf(v),$$

which is the moral value of his expectation if the shipment be in one vessel only.

(3) *In the above case, the larger the number of vessels, the greater is the moral value of his expectation.*

For if it be shipped in $(t+1)$ vessels, the moral value of his expectation is

$$p^{t+1}f(v+s)+(t+1)p^tqf\left(v+\frac{t}{t+1}s\right)+\frac{(t+1)t}{1\cdot 2}p^{t-1}q^2f\left(v+\frac{t-1}{t+1}s\right)+\cdots+q^{t+1}f(v).$$

Now $$v+\frac{t}{t+1}s=\frac{1}{t+1}(v+s)+\frac{t}{t+1}\left(v+\frac{t-1}{t}s\right),$$

$$v+\frac{t-1}{t+1}s=\frac{2}{t+1}\left(v+\frac{t-1}{t}s\right)+\frac{t-1}{t+1}\left(v+\frac{t-2}{t}s\right),$$

etc.

Therefore this value is greater than

$$p^{t+1}f(v+s)+\left\{p^tqf(v+s)+tp^tqf\left(v+\frac{t-1}{t}s\right)\right\}$$
$$+\left\{tp^{t-1}q^2f\left(v+\frac{t-1}{t}s\right)+\frac{t(t-1)}{1\cdot2}p^{t-1}q^2f\left(v+\frac{t-2}{t}s\right)\right\}$$
$$+\cdots+\left\{(tpq^tf\left(v+\frac{1}{t}s\right)+pq^tf(v)\right\}+q^{t+1}f(v)$$
$$=p^tf(v+s)+tp^{t-1}qf\left(v+\frac{t-1}{t}s\right)+\frac{t(t-1)}{1\cdot2}f\left(v+\frac{t-2}{t}s\right)$$
$$+\cdots+q^tf(v),$$

which is the moral value if it be shipped in vessels.

4. *In the above case the merchant may advantageously pay more than the mathematical net premium to insure the cargo.*

For suppose he merely pays the net premium $qs$. After paying it his wealth is $(v+ps)$, and its moral value is $f(v+ps)$.

Now taking the general case of $t$ vessels, before insuring, the moral value of his expectation is, as above,

$$p^tf(v+s)+tp^{t-1}qf\left(v+\frac{t-1}{t}s\right)+\frac{t(t-1)}{1\cdot2}p^{t-2}q^2f\left(v+\frac{t-2}{t}s\right)$$
$$+\cdots+q^tf(v).$$

Applying the corollary, we find this to be less than

$$f\left\{p^t(v+s)+tp^{t-1}q\left(v+\frac{t-1}{t}s\right)+\frac{t(t-1)}{1\cdot2}p^{t-2}q^2\left(v+\frac{t-2}{t}s\right)\right.$$
$$\left.+\cdots+q^tv\right\}$$
$$=f\{v(p+q)^t+ps(p+q)^{t-1}\}$$
$$=f(v+ps),$$

the moral value after insuring at the net premium. He may therefore without disadvantage pay more than the net premium.

The following diagram illustrates the case where $t=1$.

In this diagram the equation of the curve $PSRQ$ is supposed to be $y=f(x)$. On the axis of $x$ points $M$, $N$, $K$ are taken, so that $OM=v$; $MN=s$; and $MK=ps$; and therefore

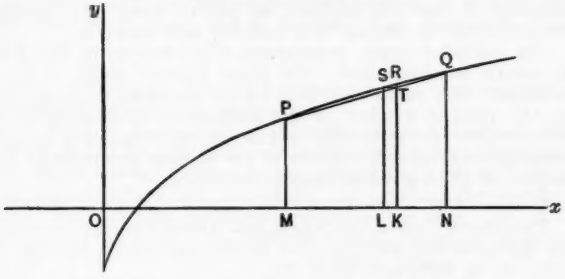
$$ON=v+s \text{ and } OK=v+ps.$$

At $M$, $N$, and $K$ ordinates are erected meeting the curve at $P$, $Q$, and $R$, respectively, so that we have

$$MP=f(v),\ NQ=f(v+s), \text{ and } KR=f(v+ps).$$

The chord $PQ$ being drawn meets $KR$ at $T$, so that

$$KT = pf(v+s) + qf(v).$$




Through $T$ draw $TS$ parallel to the axis of $x$ to meet the curve in $S$, and through $S$ draw the ordinate $SL$ meeting the axis of $x$ in the point $L$; then since $SL = TK$, we have

$OL$ = fortune in possession whose moral value equals the moral value of the expectation.

$LN$ = premium which may, without disadvantage, be paid to insure $S$, represented by $MN$.

$KN$ = net premium for same.

$LK = ST$ = loading allowed.

OTTAWA, *July* 12, 1895.

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## NOTES.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York, Saturday afternoon, October 26, at three o'clock, the President, Dr. HILL, in the chair. There were nineteen members present. On the recommendation of the Council, the following persons, nominated at the preceding meeting, were elected to membership: Professor HENRY BYRON NEWSON, University of Kansas, Lawrence, Kas.; and Professor FREDERICK SHENSTONE WOODS, Massachusetts Institute of Technology, Boston, Mass. Seven nominations for membership were received. The President announced to the Society that an invitation had been received from the President of the Northwestern University, Evanston, Ill., inviting the Society to hold its next summer meeting at

Evanston under the auspices of the Northwestern University. He stated that he had replied to the invitation on behalf of the Society, expressing its appreciation of the offer, although it was not possible as yet to make any definite arrangements in regard to a meeting next summer.

The following paper, transmitted to the Society by Mr. J. C. GLASHAN, was presented: "On Moral Values," by Mr. R. HENDERSON. The paper was read by the Secretary. It appears in the present number of the BULLETIN. Professor R. S. WOODWARD communicated a note on an error in Church's Descriptive Geometry relative to the surface generated by the motion of a straight line tangent to a helix.

PROFESSOR FRANZ NEUMANN, well known through his works on mathematical physics, died at Königsberg, on the 23d of May, at the advanced age of 96.

THE director of the Astronomische Recheninstitut and professor of astronomy in the University of Berlin, Dr. F. Tietjen, died at Berlin on the 21st of June, aged 61.

PROFESSOR FRIEDRICH GUSTAV MEHLER, who died in July, was the author of several papers published in *Crelle's Journal* and of a widely used text-book of elementary mathematics (Hauptsätze der Elementar-Mathematik). At the time of his death he was professor in the Elbing Gymnasium.

THE mathematical prize of the Prince Jablonowski Society for 1894 (1000 marks) has been awarded to Dr. Paul Harzer, director of the Gotha Observatory. The problem proposed by the Society required *a new determination of the secular perturbations of at least the orbits of Mercury, Venus, the Earth, and Mars, having regard to the terms of higher order* (see BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, Vol. I, p. 22). It appears from the Annual Report of the Prince Jablonowski Society, dated Leipzig, March, 1895, that while Dr. Harzer's memoir contains a complete solution of the theoretical part, the practical part of the problem is carried only through the preliminary calculations up to the actual integration. But as the calculations are carried sufficiently far to warrant a very favorable judgment concerning the adequacy of the method used by the author, the Society has decided to publish the memoir, urgently requesting the author to continue his work so as to effect the numerical evaluation of the secular perturbations.

THE University of the City of New York has appointed Mr. Arthur D. Frizell an assistant in mathematics.

THE organization of an "International Association for Promoting the Study of Quaternions and Allied Systems of Mathematics" has been suggested. Information in regard to the proposed society can be obtained from Messrs. P. Molenbroek, The Hague, Holland, and S. Kimura, Yale University, New Haven.

B. G. TEUBNER of Leipzig has an unusually large number of mathematical works in preparation.

To begin with the history of mathematics, the second part of Vol. III of MORITZ CANTOR'S "Vorlesungen über Geschichte der Mathematik," covering the period from 1700 to 1726, is announced. The third part, which is to conclude this standard work, will carry the historical account to the year 1759. The excellent edition, by I. L. Heiberg and H. Menge, of the complete works of Euclid ("EUCLIDIS Opera omnia," Greek text with Latin translation) of which Vols. I–V and Vol. VII have appeared before, will be brought to a close this year by the appearance of Vol. VI, which contains the "Data" edited by Dr. Menge. As the second volume of Paul Tannery's edition of Diophantus ("DIOPHANTI ALEXANDRINI Opera omnia cum Graecis commentariis") is in press, we are now in possession of most carefully prepared texts of the four classics of Greek mathematics: Euclid, Archimedes, Apollonius, and Diophantus. A complete edition of the works of Ptolemy, by Berger, Boll, Heiberg, and others, is in preparation.

In modern mathematics Teubner is also publishing several sets of collected works. The second part of Vol. I of HERMANN GRASSMANN'S mathematical writings, edited by Professor F. Engel, contains the "Ausdehnungslehre" of 1862, and will probably be out before the end of the present year. Two more volumes are to follow. The mathematical volume of JULIUS PLÜCKER'S "Wissenschaftliche Abhandlungen," edited by Professor A. Schoenflies, has just appeared; the physical volume will be edited by Dr. F. Pockels. The first volume of LEOPOLD KRONECKER'S "Vorlesungen," which appeared in 1894, has now been followed by the first volume of his "Gesammelte Werke"; each of these two series will consist of four volumes. P. STÄCKEL and F. ENGEL'S "Theorie der Parallellinien von Euklid bis auf Gauss, eine Urkundensammlung," and O. BIERMANN'S "Elemente der höheren Mathematik," appeared during the last few months. The following works are announced as in press: a German translation, by M. Lukat, of LUIGI BIANCHI'S "Geometria differenziale"; a new work by Professor CARL NEUMANN, "Allgemeine Untersuchungen über das Newton'sche Princip der Fernwirkungen"; the first volume of a "Theorie der doppelt periodischen Funktionen einer Veränderlichen," by Professor MARTIN KRAUSE; the second part of Vol. II, as

well as Vol. III, of Professor ERNST SCHRÖDER's "Vorlesungen über die Algebra der Logik"; and the long expected "Geometrie der Zahlen," by Professor HERMANN MINKOWSKI. The printing of the first part of a new work by Professor SOPHUS LIE and Dr. GEORG SCHEFFERS, "Geometrie der Berührungstransformationen," is nearly completed, and the work will probably appear in December.

Among other works announced as soon forthcoming, we may mention: L. SCHLESINGER, "Handbuch der Theorie der linearen Differentialgleichungen," Vol. II; O. STOLZ, "Grundzüge der Differential- und Integralrechnung," Vol. II; R. STURM, "Die Gebilde ersten und zweiten Grades der Liniengeometrie in synthetischer Behandlung," Vol. III; P. GORDAN, "Vorlesungen über Invariantentheorie," Vol. III; and W. WIRTINGER, "Untersuchungen über Thetafunktionen," a memoir which was awarded the Benecke prize by the Philosophical Faculty of the University of Göttingen in 1895. A work on automorphic functions, by Professor FELIX KLEIN and Dr. ROBERT FRICKE, is also in preparation.

In applied mathematics Teubner announces the second volume of the fourth edition of WÜLLNER's "Experimental-Physik"; Vol. VIII of FRANZ NEUMANN's "Vorlesungen über mathematische Physik: Vorlesungen über die Wärme"; a second part to Professor CARL NEUMANN's "Elektrische Kräfte"; Vol. II of A. STEINHEIL and E. VOIT's "Handbuch der angewandten Optik"; and MARTIN, "Die Folgerungen der Bodenreinertragstheorie" (in 3 vols.), Vol. II.

AMONG the announcements of MM. Gauthier-Villars et Fils of Paris we notice the fourth (and last) volume of G. DARBOUX's "Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal"; a first fascicule (352 pp.) of this volume has appeared. Of MÉRAY's "Leçons nouvelles sur l'analyse infinitésimale et ses applications géométriques" the second volume has been issued, while the third and fourth ("Questions analytiques classiques; applications géométriques") are in preparation. The third and last volume of FERMAT's works is in press. CAUCHY's "Œuvres complètes" are progressing slowly; vols. 1, 4, 5, 6, 7, 8 of series I, and vols. 6, 7, 8, 9, 10 of series II have appeared so far; vol. 9 of the first series is nearly ready. Eleven of the thirteen volumes of the magnificent edition of LAPLACE's works are ready; the twelfth is in press. PICARD's valuable "Traité d'analyse" is in its third volume, of which two parts (390 pp.) have appeared; the fourth and final volume, on partial differential equations, is in preparation. The second volume of J. TANNERY and MOLK's "Éléments de la théorie des fonctions elliptiques" is still in press; the whole work will comprise

four volumes. Of B. NIEWENGLOWSKI's "Cours de géométrie analytique," Vol. I was issued in 1894, Vol. II in 1895; the third ("Géométrie dans l'espace, avec une note sur les transformations en géométrie, par E. Borel") is in press. The fourth volume of TISSERAND's "Traité de mécanique céleste" has not yet appeared, but is printing; the second volume of BAILLAUD's "Cours d'astronomie" will soon be out; and a portion of the second volume of P. APPELL's "Traité de mécanique" has been issued, while the remaining part and Vol. III are in preparation.

IN the number of *Nature* for October 3 is a long list of forthcoming scientific works. Among the works announced are the following: "The year-book of scientific and learned societies of Great Britain and Ireland," comprising lists of the papers read before societies engaged in fourteen departments of research during 1895, to be published by C. Griffin & Co.; "The Indian calendar," containing complete tables for the verification of Hindu and Mohammedan dates for a period of 1600 years (A.D. 300 to 1900) for the whole of India, by ROBERT SEWELL, in collaboration with SANKARA BÂLKRISHNA DIKSHIT, with a table of eclipses by Dr. SCHRAM, to be published by Swan Sonnenschein & Co; "A practical trigonometry, for the use of engineers, architects, and surveyors," by HENRY ADAMS, to be published by Whittaker & Co.; "Miscellaneous papers," by the late Professor H. HERTZ, translated by D. E. Jones, "Elements of geometry," by GEORGE C. EDWARDS, "CHRISTIANSEN's mathematical physics," translated by W. F. Magie, "Rigid dynamics," by W. J. LOUDON, "Elements of co-ordinate geometry," by S. L. LONEY, and "Elementary trigonometry," by CHARLES PENDLEBURY, to be published by Macmillan & Co. "Dynamics," by P. G. TAIT, to be published by A. & C. Black; "An introduction to the algebra of quantics," by E. B. ELLIOTT, to be published by the Clarendon Press; and, to be published by the Cambridge University Press, "Catalogue of scientific papers, compiled by the Royal Society of London," 1874–1883, Vol. XI; "Scientific papers of JOHN COUCH ADAMS," Vol. I, edited by his brother, W. G. Adams, with a memoir by Dr. J. W. L. Glaisher; "Collected mathematical papers of ARTHUR CAYLEY," Vol. IX; "Treatise on spherical astronomy," by Sir ROBERT S. BALL; HEATH's "Treatise on geometrical optics," second edition; "Treatise on Abel's theorem," by H. F. BAKER; "Treatise on the Lunar theory," by E. W. BROWN; "Elementary treatise on electricity and magnetism," by J. J. THOMSON; "Treatise on geometrical optics," by R. A. HERMAN; "Euclid," Books XI and XII, by H. M. TAYLOR; and "Mechanics and hydrostatics; part 3, hydrostatics," by R. T. GLAZEBROOK.

THE announcements of Ginn & Co. include: "Problems in differential calculus," by W. E. BYERLY; "Molecules and the molecular theory of matter," by A. D. RISTEEN; and "Plane and solid geometry," by W. W. BEMAN and D. E. SMITH.

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## NEW PUBLICATIONS.

### HIGHER MATHEMATICS.

ABRAHAM IBN ESRA. Sefer Ha-Mispar. Das Buch der Zahl. Ein hebräisch-arithmetisches Werk aus dem 12ten Jahrhundert. Uebersetzt und erläutert von M. Silberberg. Frankfurt am Main, 1895. 8vo. 10 and 198 pp. Illustrated. Mk. 4.00

ANNUAIRE DE L'ÉCOLE POLYTECHNIQUE pour l'an 1895. Avec des notes scientifiques, historiques et statistiques et le Mémorial de l'École. Paris, 1895. 8vo. 308 pp. Mk. 2.00

BLANK (F.). Ueber die geodätischen Curven auf einem körperlichen Kreisringe. [Progr.] Gera, 1895. 4to. 24 pp. Illustrated. Mk. 1.50

CALINON (A.). Le géométrie à deux dimensions des surfaces à courbure constante. Paris, Gauthier-Villars, 1895. 8vo. Fr. 2.50

CAYLEY (A.). Collected mathematical papers. Vol. VIII. Cambridge, University Press (New York, Macmillan), 1895. 4to. 4 and 570 pp. 25s.

CHISHOLM (G.). Algebraisch-gruppentheoretische Untersuchungen zur sphärischen Trigonometrie. [Diss.] Göttingen, 1895. 8vo. 68 pp. 3 plates. Mk. 2.00

DARBOUX (G.). Leçons sur la théorie générale des surfaces et les applications géométriques du calcul infinitésimal. (In 4 vols.) Vol. IV: Déformation infiniment petite et représentation graphique. Paris, Gauthier-Villars, 1895. 8vo. 1er fasc., pp. 1–352. Whole volume Fr. 15.00

DELASSUS (E.). Sur les équations linéaires aux dérivées partielles à caractéristiques réelles. [Thèse.] Paris, Gauthier-Villars, 1895. 4to. 67 pp. Fr. 3.50

GRASSMANN (R.). Die Formenlehre der Mathematik in strenger Formelentwicklung. 4 Teile in 1 Bande, nebst Formelbuch. Stettin, Grassmann, 1895. 8vo. 5 and 12 and 242 pp.; 12 and 189 pp.; 9 and 132 pp.; 3 and 39 pp.; Formelbuch: 24, 27, 14, and 5 pp. Illustrated. Mk. 10.00

HŒFER (F.). Histoire des mathématiques depuis leurs origines jusqu'au commencement du XIXe siècle. 4e édition. Paris, Hachette, 1895. 16mo. 3 and 609 pp. Fr. 4.00

HURWITZ (J.). Ueber eine besondere Art der Kettenbruch-Entwicklung complexer Grössen. [Diss.] Halle, 1895. 8vo. 50 pp. 1 plate.

ISRAEL-HOLZWART (K.). Grundlagen und Methoden des tabellarischen Rechnens. Frankfurt am Main, 1895. 4to. 58 pp. Illustrated. Mk. 1.50

JACOBI (C. G. J.). Ueber die vierfach periodischen Functionen zweier Variabeln, auf die sich die Theorie der Abel'schen Transcendenten stützt. (1834.) Herausgegeben von H. Weber. Aus dem Lateinischen übersetzt von A. Witting. Leipzig, Engelmann, 1895. 8vo. 40 pp. Cloth. Mk. 0.70

JACOTTET (C.). Ueber die allgemeine Reihenentwicklung der Potentialfunction nach Lamé'schen Producten. [Diss.] Göttingen, 1895. 8vo. 48 pp. Illustrated. Mk. 1.20

KOVALEVSKY (S.). See LEFFLER (A. C.).

KOVALEWSKY (S.). Souvenirs d'enfance de Sophie Kovalewsky, écrits par elle-même, et suivis de sa biographie par M^me^ A. Ch. Leffler, duchesse de Cajanello. Paris, Hachette, 1895. 16mo. 10 and 334 pp. Fr. 3.50

KRONECKER (L.). Werke, herausgegeben auf Veranlassung der Königl. Preussischen Akademie der Wissenschaften von K. Hensel. (In 4 vols.) Vol. I. Leipzig, Teubner, 1895. 4to. 10 and 484 pp. 1 portrait. Mk. 28.00

KLEIN (F.). Ueber die Beziehungen der neueren Mathematik zu den Anwendungen. Antrittsrede, gehalten am 25. Oktober 1880 bei Uebernahme der damals an der Universität Leipzig neuerrichteten Professur für Geometrie. (Besonderer Abdruck aus Band XXVI der *Zeitschrift für mathematisch-naturwissenschaftlichen Unterricht.*) Leipzig, Teubner, 1895. 8vo. 12 pp.

LAPLACE. Œuvres complètes, publiées sous les auspices de l'Académie des sciences par les secrétaires perpétuels, avec le concours de MM. Puiseux, F. Tisserand, J. Hoüel et Souillard. (In 13 vols.) Vol. XI: Mémoires extraits des recueils de l'Académie des sciences. Paris, Gauthier-Villars, 1895. 4to. Fr. 20.00

LEFFLER (A. C.). Sonya Kovalevsky: a biography. By Anna Carlotta Leffler, duchess of Cajanello, and sisters Rajevsky: being an account of her life, by Sonya Kovalevsky. Translated by A. de Furuhjelm and A. M. Clive Bayley, with a biographical note by Lily Wolffsohn. London, Unwin, 1895. 8vo. 380 pp. 6s.

——— Sonia Kovalevsky: biography and autobiography. I: Memoir by by A. C. Leffler (Edgren), duchessa di Cajanello. II: Reminiscences of childhood, written by herself. Translated into English by Louise von Cossel. London, Scott (New York, Macmillan), 1895. 12mo. 316 pp. Frontispiece. $1.25

LJUNGH (A. T.). Ueber isoptische und orthoptische Kurven. [Diss.] Lund, 1895. 8vo. 50 pp. 1 plate.

LONEY (S. L.). The elements of co-ordinate geometry. London and New York, Macmillan, 1895. 12mo. 10 and 420 pp. Cloth. $1.25

MANAIRA (A.). Sopra una certa rappresentazione piana dell' elissoide di rivoluzione e sulla applicazione di essa ai calcoli geodetici. Padova, Prosperini, 1895. 8vo. 56 pp. 1 plate.

MATHEMATICAL QUESTIONS AND SOLUTIONS. From the *Educational Times*, etc. Edited by W. J. C. Miller. Vol. 63. London, Hodgson, 1895. 8vo. 128 pp. 6s. 6d.

NIELSEN (N.). Om en Klasse bestemte Integraler og nogle derved definerede semiperiodiske Funktioner. Kjöbenhavn, 1895. 4to. 80 pp. Mk. 4.00

PETERSEN (J.). Forelæsninger over Funktionsteori. Hæfte II og III. Kjöbenhavn, 1895. 8vo. 112 and 144 pp. Mk. 7.60

PLÜCKER (J.). Gesammelte wissenschaftliche Abhandlungen. Im Auftrag der Königl. Gesellschaft der Wissenschaften zu Göttingen herausgegeben von A. Schoenflies und F. Pockels. Vol. I: Mathematische Abhandlungen, herausgegeben von A. Schoenflies. Leipzig, Teubner, 1895. 8vo. 36 and 620 pp. 1 portrait. Mk. 20.00

REBSTEIN (J.). Bestimmung aller reellen Minimalflächen, die eine Schaar ebener Kurven enthalten, denen auf der Gauss'schen Kugel die Meridiane entsprechen. Berlin, 1895. 8vo. 30 pp. Mk. 1.20

RIEMANN (B.). Sur le nombre des nombres premiers inférieurs à une grandeur donnée. Traduit par L. Laugel. Paris, Gauthier-Villars, 1895. 8vo. Fr. 0.75

ROSENHAIN (G.). Abhandlungen über die Functionen zweier Variabler mit vier Perioden, welche die Inversen sind der ultra-elliptischen Integrale erster Klasse. (1851.) Herausgegeben von H. Weber. Aus dem Französischen übersetzt von A. Witting. Leipzig, Engelmann, 1895. 8vo. 94 pp. Cloth. Mk. 1.50

SCHLÖMILCH (O.). Compendium der höheren Analysis. Band II: Vorlesungen über einzelne Theile der höheren Analysis. 4te Auflage. Braunschweig, Vieweg, 1895. 8vo. 10 and 546 pp. Illustrated. Mk. 9.00

SIEVERT (H.). Ueber Thetafunktionen, deren Charakteristiken aus Fünfteln ganzer Zahlen bestehen. II. [Progr.] Bayreuth, 1895. 8vo. 36 pp.

SILBERBERG (M.). See ABRAHAM IBN ESRA.

SNYDER (V.). Ueber die linearen Complexe der Lie'schen Kugelgeometrie. [Diss.] Göttingen, 1895. 8vo. 48 pp. Illustrated. Mk. 1.40

WEBER (E. VON). Die singulären Lösungen der partiellen Differentialgleichungen mit drei Variablen. [Habilitationsschrift.] München, 1894. 8vo. 30 pp.

WEIERSTRASS (K.) Mathematische Werke. Herausgegeben unter Mitwirkung einer von der Königl. Preussischen Akademie der Wissenchaften eingesetzten Commission. Vol. II: Abhandlungen II. Berlin, Mayer & Müller, 1895. 4to. Mk. 21.00

WELD (L. G.). A short course in the theory of determinants. 12mo. 238 pp. $1.90

ZEUTHEN (H. G.). Anvendelse af Differentialregning paa algebraiske Ligninger. Kjöbenhavn, 1895. 4to. 24 pp. [Lithogr.] Mk. 0.80

## ELEMENTARY MATHEMATICS.

ALEU (M. L.) Elementos de matemáticas. Geometria. Valdemoro, 1895. 4to. 12 and 206 pp. Illustrated. Mk. 6.50

——— Elementos de matemáticas. Trigonometria rectilinea. Madrid, 1895. 4to. 52 pp. Illustrated. Mk. 2.50

ANDRIANI (A.). Elementi di algebra ad uso dei licei e degli istituti tecnici secondo gli ultimi programmi con numerosa raccolta di esercizi. 2a edizione migliorata. Napoli, Trani, 1895. 16mo. 390 pp. Fr. 4.00

BARDEY (E.). Arithmetische Aufgaben, nebst Lehrbuch der Arithmetik. 9te Auflage. Leipzig, Teubner, 1895. 8vo. 12 and 270 pp. Cloth. Mk. 2.40

——— Methodisch geordnete Aufgabensammlung, mehr als 8000 Aufgaben enthaltend über alle Theile der Elementar-Arithmetik. 21ste Auflage. Leipzig, Teubner, 1895. 8vo. 14 and 330 pp. Cloth. Mk. 3.20

BORK (H.). Periodische Decimalbrüche. [Progr.] Berlin, 1895. 4to. 42 pp.

BREUER (P. J.). Das Notwendigste über die natürlichen Logarithmen. Leipzig, Teubner, 1895. 8vo. 40 pp. Mk. 0.80

BROOKSMITH (E. J.). See WOOLWICH MATHEMATICAL PAPERS.

CAPELLI (A.). Lezioni di algebra complementare ad uso degli aspiranti alla licenza universitaria in scienze fisiche e matematiche. Napoli, Pellerano, 1895. 8vo. 12 and 526 pp. Illustrated. Fr. 8.00

CIAMBERLINI (C.). Elementi di geometria, ad uso delle scuole tecniche, normali e professionali. Parte I: Planimetria. Montegiorgio, Delbello, 1895. 16mo. 149 pp.

COMINOTTO (E.). Trisezione approssimata dell' angolo. Padova, Tip. del Seminario, 1895. 8vo. 7 pp.

——— Sopra una disposizione particolare dei triangoli simili: seguito di una nota che verrà pubblicata nel periodico *Lugli* di Roma. Padova, Tip. del Seminario, 1895. 8vo. 7 pp.

FRANKENBACH (W.). Die Harmonikalen der Mittelpunkte der Berührungskreise eines Dreiecks in Bezug auf dasselbe. Liegnitz, 1895. 8vo. 32 pp. Illustrated. Mk. 1.50.

GAUSS (F. G.). Fünfstellige vollständige logarithmische und trigonometrische Tafeln. 46ste Auflage. Halle, 1895. 8vo. 166 and 35 pp. Cloth. Mk. 2.50

GÉRARD (L.). Géométrie à l'usage des candidats aux baccalauréats de l'enseignement secondaire. Paris, 1895. 8vo. Fr. 3.00

GREMIGNI (M.). La teorica dei poligoni equivalenti. Firenze, Paggi, 1895. 8vo. 28 pp. Fr. 1.00

HOLZMÜLLER (G.). Methodisches Lehrbuch der Elementar-Mathematik. Im engsten Anschluss an die neuen Lehrpläne. (In 3 Teilen.) Teil I. 2te Auflage. Leipzig, Teubner, 1895. 8vo. 8 and 230 pp. Cloth. Mk. 2.40

LAISANT (C. A.) et LEMOINE (E.). Traité d'aritmétique; suivi de notes sur l'ortografie simplifiée, par P. Malvezin. Paris, Gauthier-Villars, 1895. 16mo. 8 and 174 pp.

LAISANT (C. A.). Recueil de problèmes de mathématiques, classés par divisions scientifiques, contenant les énoncés avec renvoi aux solutions de tous les problèmes posés, depuis l'origine, dans divers journaux: *Nouvelles annales de mathématiques, Journal de mathématiques élémentaires et de mathématiques spéciales, Nouvelle correspondance mathématique, Mathésis.* (In 7 vols.) Vol. III: Algèbre; théorie des nombres; probabilités; géométrie de situation. Paris, Gauthier-Villars, 1895. 8vo. Fr. 6.00

LEGENDRE. Elementi di geometria, con giunte e modificazioni di A. Blanchet. Versione italiana, conforme ai programmi ministeriali, autorizzata ed approvata da A. Blanchet, con note ed aggiunte di G. Hueber. 26ma edizione. Napoli, Morano, 1894. 16mo. 310 pp. Illustrated. Fr. 4.00

LEMOINE (E.). See LAISANT (C. A.) et LEMOINE (E.).

LESKY (A.). Bestimmung der Maxima und Minima quadratischer Gleichungen. Klagenfurt, 1895. 8vo. 20 pp. Mk. 1.00

LIVINI (A.). Elementi di geometria e topografia. Vol I: Geometria piana e stereometria. 3a edizione. Napoli, Pontieri, 1894. 16mo. 140 pp. Illustrated. Fr. 2.00

LOMHOLT (A.). Regulære Polyedre. En elementær Skitse. Viborg, 1895. 8vo. 30 pp. 2 plates. Mk. 0.70

LUCAS (É.). L'arithmétique amusante. Amusements scientifiques pour l'enseignement et la pratique du calcul. (Introduction aux *Récréations mathématiques.*) Paris, Gauthier-Villars, 1895. 8vo. Fr. 7.50

MALAGODI (A.). Nozioni d'algebra elementare, con numerosi esercizi e problemi, ad uso della terza classe delle scuole tecniche. 2a edizione riveduta e corretta. Mirandola, Cagarelli, 1895. 8vo. 64 pp. Fr. 0.50

MARTI (R.). Operazioni del calcolo e loro analogia. Lecce, Tip. cooperativa, 1895. 8vo. 36 pp.

MONNET (L. F.). Cours élémentaire d'algèbre et de trigonométrie théorique et pratique, à l'usage des lycées, des collèges, etc. 11e édition. Paris, André, 1894. 18mo. 324 pp. Illustrated.

——— Cours élémentaire de géométrie théorique et pratique, à l'usage des lycées, des collèges, etc., contenant un très grand nombre d'exercices résolus et à résoudre. 15e édition. Paris, André, 1895. 18mo. 360 pp.

MONTI (P.). Regola generale per la soluzione grafica della trisezione dell' angolo, colla dimostrazione di G. Schiaparelli. Milano, Bombelli, 1895. 4to. 7 pp. 1 plate.

MOSER (J.). Harmonische Strahlen und Punkte. Breslau, 1895. 4to. 12 pp. Illustrated. Mk. 1.20

ORTU CARBONI (S.). Geometria descrittiva elementare ed alcune sue applicazioni (proiezioni ortogonali). Vol. II. Torino, Paravia, 1895. 8vo. 4 and 158 pp. Illustrated. Fr. 3.00

OTTO (F. A.). Die rechnende Geometrie (System Otto), eine unerschöpfliche Fundgrube für mathematische Forschung. Mit Anhang: Das geometrische Problem des 20sten Jahrhunderts (Otto's Problem). Tegel, 1895. 12mo. 36 pp. Illustrated. Mk. 1.00

——— Ein Problem der Rechenkunst. Rationelle Methode zur Auflösung von numerischen Gleichungen mit einer Unbekannten (beliebiger Grad und jede Form). 2te Auflage. Tegel, 1895. 12mo. 32 pp. Mk. 0.50

PERSIANI (O.). Elementi di geometria secondo Euclide, ad uso dei licei. Libri V–VI. Roma, Spellani, 1895. 8vo. 52 pp. Illustrated.

PESCI (G.). Trattato elementare di trigonometria piana e sferica; con appendice. Livorno, 1895. 8vo. 330 and 80 pp. Illustrated. Fr. 5.00

PIETZKER (F.). See UNTERRICHTSBLÄTTER.

RUSSELL (J. W.). An elementary treatise on pure geometry. With numerous examples. 12mo. 324 pp. $2.60

SCHWALBE (B.). See UNTERRICHTSBLÄTTER.

STROMILLO (S.). Lezioni elementari di geometria per le scuole secondarie. Parti I e II. Napoli, Muca, 1894. 16mo. 107 and 135 pp. Illustrated. Fr. 4.25

THOMPSON (W. T.). Scholarship algebra. London, Hughes, 1895. 8vo. 280 pp. 2s. 6d.

UNTERRICHTSBLÄTTER für Mathematik und Naturwissenschaften. Herausgegeben von B. Schwalbe und F. Pietzker. Braunschweig, Vieweg, 1895. 4to. 5 numbers per year. Mk. 3.00

VINE (G. T.). Mathematical note-book, specially prepared for the use of pupils, teachers, scholarship and certificate candidates. Newport, 1895. 8vo. 88 pp. Cloth. $0.50

WEEKS (W.). Exercises in Euclid, graduated and systematized. 18mo. 80 pp. $0.60

WOOLWICH MATHEMATICAL PAPERS for admission into the Royal Military Academy, for the years 1885–94. Edited by E. J. Brooksmith. London and New York, Macmillan, 1895. 12mo. Cloth. $1.50

## APPLIED MATHEMATICS.

ALBRICH (K.). Der Unterricht in Mechanik auf geschichtlicher Grundlage. Hermannstadt, 1894. 4to. 35 pp. Illustrated. Mk. 1.50

ALHEILIG et ROCHE (C.). Traité des machines à vapeur, rédigé conformément au programme du cours de machines à vapeur de l'École centrale. (In 2 vols.) Vol. I. Paris, Gauthier-Villars, 1895. 8vo. 10 and 604 pp. Illustrated. Fr. 20.00

ASSMANN (R.). See FORTSCHRITTE DER PHYSIK.

BERTOLDO (G.). Compendio di termodinamica applicata. Vol. I. Torino, Unione tipografico-editrice, 1895. 8vo. 239 pp. Illustrated. Fr. 4.00

BÖRNSTEIN (R.). See FORTSCHRITTE DER PHYSIK.

BOURLET (C.). Traité des bicycles et des bicyclettes, suivi d'une application à la construction des vélodromes. (Encyclopédie scientifique des aide-mémoire.) Paris, Gauthier-Villars, 1895. 8vo. Cloth. Fr. 3.00

BROWN (H. T.). 500 meccanismi, scelti fra i più importanti e recenti, riferentisi alla dinamica, idraulica, idrostatica, pneumatica, macchine à vapore, molini, torchi, orologeria ed altre macchine. 2a edizione, compilata sulla 16a edizione inglese da F. Cerruti. Milano, 1895. 12mo. 182 pp. Illustrated. Fr. 3.00

CART (A.) et PORTES (L.). Calcul des ponts métalliques à poutres droites, à une ou plusieurs travées, par la méthode des lignes d'influence. Formules et tables. Paris, Baudry, 1895. 8vo. 8 and 292 pp.

CAZES (L.). Stéréoscopie de précision: théorie et pratique. Paris, 1895. 8vo. 72 pp. Illustrated. Fr. 2.50

CERRUTI (F.). See BROWN (H. T.).

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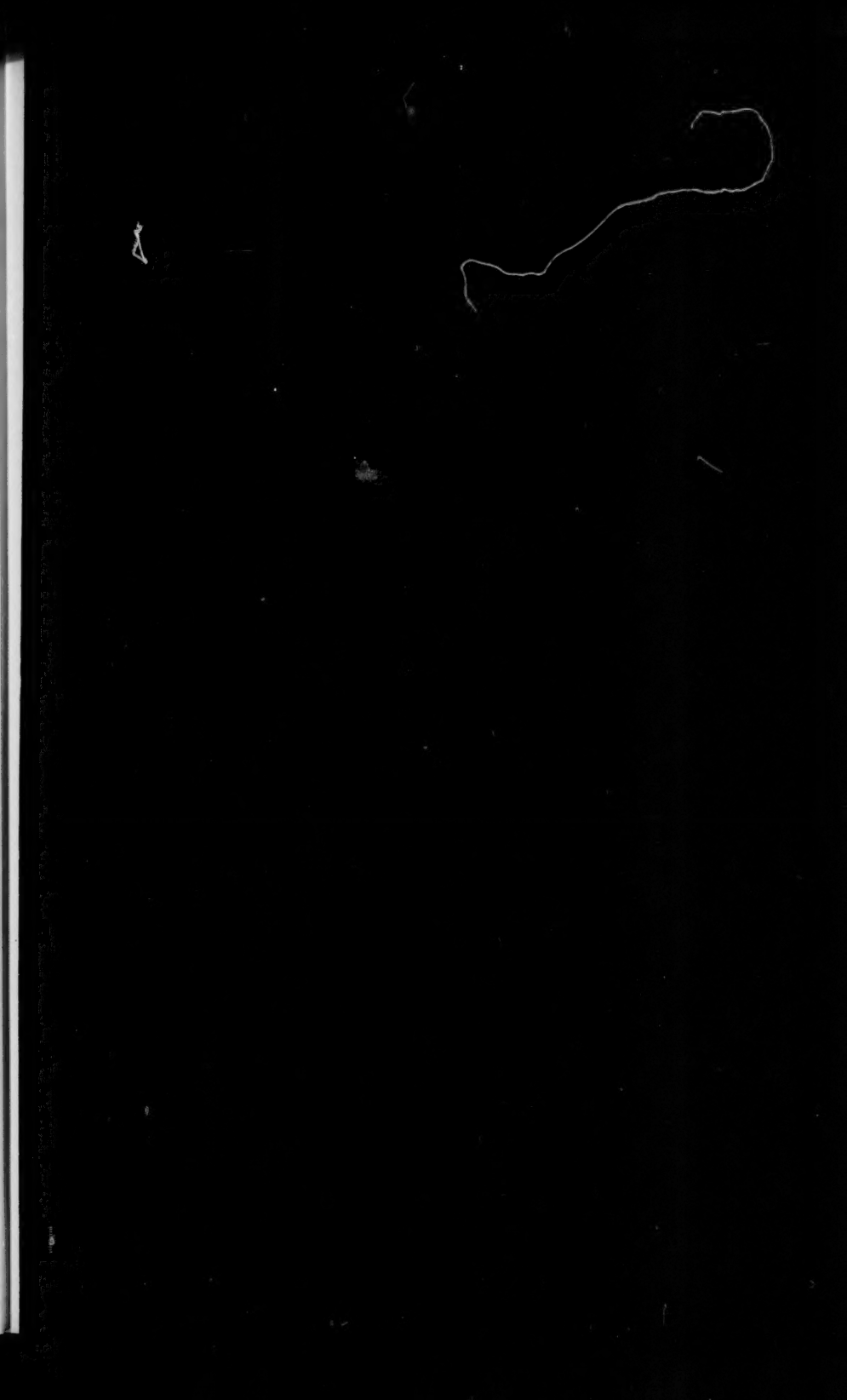
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